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Inverse scattering with supersymmetric quantum mechanics

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Abstract

The application of supersymmetric quantum mechanics to the inverse scattering problem is reviewed. The main difference with standard treatments of the inverse problem lies in the simple and natural extension to potentials with singularities at the origin and with a Coulomb behaviour at infinity. The most general form of potentials which are phase-equivalent to a given potential is discussed. The use of singular potentials allows adding or removing states from the bound spectrum without contradicting the Levinson theorem. Physical applications of phase-equivalent potentials in nuclear reactions and in three-body systems are described. Derivation of a potential from the phase shift at fixed orbital momentum can also be performed with the supersymmetric inversion by using a Bargmann-type approximation of the scattering matrix or phase shift. A unique singular potential without bound states can be obtained from any phase shift. A limited number of bound states depending on the singularity can then be added. This inversion procedure is illustrated with nucleon–nucleon scattering.

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1. Introduction

Inverse problems appear in various fields of physics [1]. They consist in deriving the input of some theory or model from experimental or simulated data. In particular, the inverse scattering problem in nonrelativistic quantum mechanics consists in deducing a potential from scattering information [1–3].

Ideally the potential should be determined from real experimental data, i.e. from measured cross sections. This is realized mostly in a very simplified version of the inversion procedure: some potential form factor is selected for physical reasons and its parameters are fitted to the data. This naive inversion technique can be improved to a fit of a potential expanded on some

selected basis. However, these techniques suffer from a lack of unicity and several families of potentials can usually be obtained. More elaborate numerical techniques based on cross sections or other data at different energies are available [4].

The theory of inverse scattering requires some intermediate step, i.e. a phase-shift analysis. The experimental cross sections are first expressed as a function of a set of phase shifts or of scattering matrices if more than one channel is open. This analysis is difficult and requires accurate cross sections over a broad energy range. The existence of several energies allows in principle elimination of ambiguities in phase-shift determinations. Such analyses have only been performed for a few systems of light nuclei, the most elaborate ones being without any doubt the analyses of the proton–proton and neutron–proton scatterings.

In the following, we assume that the collision is described in nonrelativistic quantum mechanics with an unknown central potential. We also assume that the phase shifts related to this potential are available. Then the inverse problem can be separated into two main types: inverse problem at fixed orbital momentum and inverse problem at fixed energy. Although the second problem is more frequent in practice because of the difficulty of obtaining data at many energies, the first problem is better posed in mathematical terms. In the rest of this review, we mainly address the inverse problem at fixed orbital momentum. This means that we assume that, for a partial wave at given orbital momentum, the phase shift is available over the full energy domain from zero to infinity. This assumption is of course impossible to satisfy since new channels always open when the energy increases and Schrödinger quantum mechanics stop being valid at some energy. Nevertheless, with this assumption, inverse scattering becomes a well-posed mathematical problem.

The inverse problem at fixed orbital momentum does not possess a unique solution as was first believed. The phase shifts do not contain information about bound states except maybe about their number through the Levinson theorem [1, 2]. This was first realized by Bargmann [5, 6]. Even the knowledge of the phase shift and of the bound-state energies does not yield a unique solution. With each bound state is associated a free parameter which is usually named 'normalization constant' [1, 2]. Therefore phase-equivalent potentials exist, i.e. different potentials with exactly the same phase shift for a given partial wave [6, 7]. Only potentials without any bound state are uniquely defined from the phase shift.

A rigorous theory of the inverse problem at fixed orbital momentum has been available for some time. It is based on the Gel'fand-Levitan or Marchenko equations [1–3, 8, 9] (see the appendix). In the present review we focus on an approach of the inverse problem based on supersymmetric quantum mechanics [10–14]. This more recent approach has two main advantages. Although it does not improve the traditional approach (it is in fact less general), it is much simpler to deal with. Moreover, the supersymmetric approach opened novel physical views on the problem by naturally introducing the use of singular potentials [15, 16]. In the following, we successively consider two aspects of inverse scattering: the construction of phase-equivalent potentials and inversion at fixed orbital momentum. We also briefly address the problem at fixed energy where supersymmetric quantum mechanics also offers interesting perspectives.

Supersymmetric quantum mechanics provide a simple general solution to the problem of phase equivalence. This was first noticed by Sukumar who derived phase-equivalent potentials in cases where the spectrum remains unchanged [11]. More generally, phase-equivalent potentials with modified spectra can be exactly derived with pairs of supersymmetric transformations [15–20]. Bound states can be added or eliminated from the bound spectrum of a potential without affecting its phase shifts. Note that such modifications do not affect the value of the orbital momentum as was sometimes stated [11] because the angular part of wavefunctions is not affected by such transformations. Another incorrect statement in initial

formulations [15–17] is that this technique can only be applied to transformations involving nodeless wavefunctions. This natural restriction for the properties of single transformations [10, 11] is in fact not necessary for pairs of transformations as was proved in [18]. This allowed us to derive the most general form of phase-equivalent potentials [20]. Equivalent potentials providing arbitrary modifications of the bound spectrum can be constructed provided that the number of added bound states does not exceed some limit fixed by the properties of the original potential.

This helped clarifying the long-standing problem of the deep or shallow nature of the nucleus–nucleus potentials. In shallow potentials all bound states have a physical meaning. Microscopic models, however, suggest that realistic potentials should be deep. Deep potentials contain unphysical 'forbidden' states in order to simulate the Pauli principle between the interacting particles. Elastic scattering data do provide both deep and shallow potentials. Supersymmetric transformations of deep potentials into shallow phase-equivalent ones show that this is a false problem for elastic scattering [15]. This does not mean that deep and shallow potentials have the same physical interest. Off-shell properties of phase-equivalent potentials are different and this may have physical consequences in reactions and in many-body systems.

Inverse scattering for a given partial wave can be performed in a simple approximate way with supersymmetric transformations [21]. This approach is equivalent to a particular case of standard inverse-scattering theory, based on a Bargmann-type approximation [5, 6] of the scattering matrix or phase shifts. The natural use of singular potentials in supersymmetric quantum mechanics suggests dividing the problem into two steps. Indeed a unique singular potential without bound state can be derived from any set of phase shifts. Bound states can subsequently be added in the most appropriate way. In particular, bound-state additions can be used to derive potentials with the weakest possible dependence on the orbital momentum [22].

The principle of supersymmetric transformations is summarized in section 2. Phase equivalence is analysed in section 3. Physical applications are described in section 4. Different facets of the inversion problem are discussed in section 5 and illustrated in section 6. Concluding remarks and an outlook are presented in section 7.

2. Types of supersymmetric transformations

This section follows Sukumar's paper [11]. The radial Schrödinger equation of a system of two particles is written as

$$H_0\varphi_0 = E\varphi_0 \tag{2.1}$$

with the Hamiltonian

$$H_0 = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_0(r). \tag{2.2}$$

We choose $\hbar = 2\mu = e = 1$ where μ is the reduced mass of the particles. Here and in the following, the subscript refers to some Hamiltonian obtained after a number of transformations. Subscript 0 refers to the initial Hamiltonian.

The Coulomb and centrifugal terms are included in potential V_0,

$$V_0(r) \underset{r\to\infty}{\to} Z_1Z_2r^{-1} + l(l+1)r^{-2} + O(r^{-3}) \tag{2.3}$$

where Z_1 and Z_2 are the charges of the colliding particles and l is the orbital momentum of their relative motion. Dependences on l will in general be implied. The potential V_0 is allowed to be singular at the origin in the following way,

$$V_0(r) \underset{r\to 0}{\to} n_0(n_0+1)r^{-2} + O(r^{-1}) \tag{2.4}$$

where n_0 is a non-negative integer. The singularity parameter n_0 may thus differ from the orbital momentum l which controls the asymptotic expression (2.3).

Let us denote as $\varphi_0(r)$ a real solution of the Schrödinger differential equation at some arbitrary energy E. This solution may represent a physical bound state when E belongs to the bound spectrum of H_0 (in that case, it is assumed to be normalized to unity) or a scattering wavefunction when E is positive. It may also represent any non-physical solution of (2.1).

The possible asymptotic behaviours of a solution at energy $E < 0$ read

$$\varphi_0(r) \underset{r\to\infty}{\to} \exp[-\sqrt{-E}r - (Z_1Z_2/2\sqrt{-E})\ln 2\sqrt{-E}r] \tag{2.5}$$

or

$$\varphi_0(r) \underset{r\to\infty}{\to} \exp[\sqrt{-E}r + (Z_1Z_2/2\sqrt{-E})\ln 2\sqrt{-E}r]. \tag{2.6}$$

For simplicity, in most of the following, we do not display the logarithmic corrections due to the Coulomb term in (2.3). However, all the presented properties do apply to potentials with a Coulomb asymptotic form. At the origin, the singularity (2.4) of the potential allows the behaviours

$$\varphi_0(r) \underset{r\to 0}{\to} r^{n_0+1} \tag{2.7}$$

or

$$\varphi_0(r) \underset{r\to 0}{\to} r^{-n_0}. \tag{2.8}$$

Function φ_0 may represent non-physical solutions which do not vanish at the origin. Until now, we do not eliminate any possible mathematical solution. Nevertheless physical wavefunctions (denoted as ψ_0) are restricted by the fact that they must be bounded. Solutions with the behaviour (2.8) and $n_0 \geqslant 0$ are called singular because the three-dimensional wavefunction $r^{-1}\varphi_0(r)Y_l^m(\theta,\varphi)$ is not bounded. Physical wavefunctions of H_0 must behave as r^{n_0+1} near the origin. For a negative energy E they must also satisfy (2.5). At a positive energy E, they behave as

$$\psi_0(r) \underset{r\to\infty}{\to} \sin\left[kr - \tfrac{1}{2}l\pi - (Z_1Z_2/2k)\ln 2kr + \delta\right] \tag{2.9}$$

where $k = \sqrt{E}$ is the wavenumber and δ is the phase shift.

The validity of several previous expressions is restricted by a condition on the potential. Phase shifts are obtained from an analysis of cross sections. This analysis is performed under the assumption that the potential tends for large r values towards the Coulomb potential, the difference between them decreasing faster than r^{-2}. The r^{-2} term in (2.3) totally originates from the centrifugal kinetic energy. The asymptotic expression (2.3) of each effective potential V_0 is therefore linked with the orbital momentum l of the corresponding partial wave, i.e. with its angular dependence. Under this assumption, the corresponding phase shift is defined by the asymptotic expression (2.9). Before defining transformations of (2.1), let us emphasize that they apply to this radial equation only. They do not affect the angular symmetry of a wavefunction. In other words, the orbital momentum l cannot vary in such transformations.

We consider some arbitrary energy $\mathcal{E} < 0$, called the factorization energy, and the corresponding solution φ_0 of the Schrödinger equation (2.1). The initial Hamiltonian H_0 can be factorized [10] as

$$H_0 = A_0^+A_0^- + \mathcal{E} \tag{2.10}$$

with the mutually adjoint first-order differential operators

$$A_0^\pm = \pm\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\mathrm{d}}{\mathrm{d}r}\ln\varphi_0. \tag{2.11}$$

The derivative of the logarithm of φ_0 is a convenient notation for φ_0'/φ_0. Therefore, one does not need to worry about the sign or normalization of φ_0. The solution φ_0 at energy $\mathcal{E}$ can change sign and be physical or non-physical.

The supersymmetric partner of H_0 reads

$$H_1 = A_0^- A_0^+ + \mathcal{E} \tag{2.12}$$

$$= -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_1(r). \tag{2.13}$$

The corresponding potential is given by

$$V_1 = V_0 - 2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\varphi_0. \tag{2.14}$$

The Hamiltonian H_1 possesses the same bound spectrum as H_0, with the possible exception of $\mathcal{E}$, when $\mathcal{E}$ is the energy of a bound state. In this particular case, a state with energy $\mathcal{E}$ is suppressed when φ_0 is a physical (bounded) solution or can be added; otherwise, the bound spectrum remains unchanged [10]. These properties can easily be observed on the wavefunctions as shown below.

Potential V_1 is singular at the zeros of φ_0, if φ_0 has nodes. In order to have a non-singular potential at finite distances, φ_0 must be nodeless, i.e. $\mathcal{E}$ must be lower than or equal to the ground-state energy of H_0. This 'no-node' condition imposed to supersymmetric transformations [10, 15–17] is, however, not mandatory when pairs of transformations are considered (see section 3).

Let us detail the properties of V_1 at 0 and ∞. The asymptotic behaviour of potential V_1 is

$$V_1(r) \underset{r\to\infty}{\to} Z_1 Z_2 r^{-1} + [l(l+1) \pm (Z_1 Z_2/\sqrt{-\mathcal{E}})]r^{-2} + O(r^{-3}). \tag{2.15}$$

The centrifugal term is modified unphysically when both particles are charged. The $-$ and $+$ signs correspond to (2.5) and (2.6), respectively. At the origin, the singularity is modified depending on the behaviour of φ_0,

$$V_1(r) \underset{r\to 0}{\to} n_1(n_1+1)r^{-2} \tag{2.16}$$

with $n_1 = n_0 + 1$ if $\varphi_0 \underset{r\to 0}{\to} r^{n_0+1}$ and $n_1 = n_0 - 1$ if $\varphi_0 \underset{r\to 0}{\to} r^{-n_0}$. A solution regular at the origin increases the singularity parameter while a solution singular at the origin decreases it. Note, however, that the orbital momentum l remains unchanged since it is fixed by the angular part of the wavefunctions. When $n_0 = 0$ and φ_0 does not vanish at the origin as in (2.8), the supersymmetric transformation does not introduce a singularity in V_1, i.e. $n_1 = 0$.

The solutions (physical or non-physical) of H_1 are related to those of H_0 by

$$\psi_1 = A_0^- \psi_0 \tag{2.17}$$

or

$$\psi_1 = (\varphi_0)^{-1} W(\psi_0, \varphi_0) \tag{2.18}$$

where $W(\psi_0, \varphi_0) = \psi_0\varphi_0' - \psi_0'\varphi_0$ is the Wronskian of ψ_0 and φ_0. For $E \neq \mathcal{E}$, these equations transform regular solutions of H_0 into regular solutions of H_1 and singular solutions of H_0 into singular solutions of H_1. They give thus access to all the solutions of H_1. However, when $n_0 = 0$ and φ_0 is singular at the origin, the nature of the solution is not always conserved by the transformation. Since this case is not useful for the applications presented later, we shall not consider it in the following.

Using an integral expression of this Wronskian, one obtains up to a constant factor

$$\psi_1 = (\varphi_0)^{-1}\left(\beta + \int_{r_0}^{r} \varphi_0\psi_0\, dt\right). \tag{2.19}$$

Table 1. Modifications of the bound spectrum, phase shift and singularity parameter for the different types of transformations (E_0 represents a bound-state energy).

Action	$\mathcal{E}$	$\lim_{r\to 0}\varphi_0$	$\lim_{r\to\infty}\varphi_0$	$\delta_1(k)-\delta_0(k)$	n_1
Suppression	$=E_0$	r^{n_0+1}	$\exp(-\sqrt{-\mathcal{E}}r)$	$\arctan(k/\sqrt{-\mathcal{E}})$	n_0+1
Addition ($n_0>0$)	$\neq E_0$	r^{-n_0}	$\exp(\sqrt{-\mathcal{E}}r)$	$-\arctan(k/\sqrt{-\mathcal{E}})$	n_0-1
None	$\neq E_0$	r^{n_0+1}	$\exp(\sqrt{-\mathcal{E}}r)$	$-\arctan(k/\sqrt{-\mathcal{E}})$	n_0+1
None ($n_0>0$)	$\neq E_0$	r^{-n_0}	$\exp(-\sqrt{-\mathcal{E}}r)$	$\arctan(k/\sqrt{-\mathcal{E}})$	n_0-1

The real parameters β and $r_0 \geqslant 0$ are somewhat redundant but are both useful to allow the different types of behaviours of φ_0 and ψ_0 in a general solution. This expression can also be verified directly. For $E=\mathcal{E}$, (2.18) only leads to one solution of H_1, i.e. $(\varphi_0)^{-1}$. At this energy, it can be verified that (2.19) with $\psi_0=\varphi_0$ and an arbitrary β still provides the general solution of H_1.

Equation (2.17) leads to the asymptotic behaviour of wavefunctions with $E>0$,

$$\psi_1(r) \underset{r\to\infty}{\to} \sin\left(kr-\tfrac{1}{2}l\pi+\delta\pm\arctan(k/\sqrt{-\mathcal{E}})\right). \tag{2.20}$$

The positive sign occurs if $\varphi_0(r)\underset{r\to\infty}{\to}\exp(-\sqrt{-\mathcal{E}}r)$ and the negative sign if $\varphi_0(r)\underset{r\to\infty}{\to}\exp(+\sqrt{-\mathcal{E}}r)$. Equation (2.20) shows that the phase shifts are modified in a simple way. This is not true in the excluded $n_0=0$ cases where positive-energy wavefunctions are transformed into non-physical states.

When $\mathcal{E}$ is the energy of a bound state of H_0 and φ_0 is its wavefunction, the transformation suppresses this state from the bound spectrum. Indeed, (2.19) with $\psi_0=\varphi_0$, $r_0=\infty$ and $\beta=0$ provides a solution of H_1 vanishing at infinity but singular at the origin. In contrast, when $\mathcal{E}$ does not belong to the bound spectrum of H_0, three cases may occur. When φ_0 is regular at the origin and singular at infinity (respectively singular at the origin and regular at infinity), $(\varphi_0)^{-1}$ is a solution of H_1 which is singular at the origin and regular at infinity (respectively regular at the origin if $n_0>0$ and singular at infinity). Hence, $\mathcal{E}$ does not belong to the spectrum of H_1 and these two transformations do not modify the bound spectrum. When φ_0 is singular both at the origin and infinity, $\psi_1=(\varphi_0)^{-1}$ is regular both at the origin and infinity. It is thus a bound-state wavefunction and this transformation adds a state to the bound spectrum. However, this is not true in the excluded $n_0=0$ case.

In summary, four types of supersymmetric transformations are obtained depending on the choice of factorization energy $\mathcal{E}$ and on the choice of solution φ_0. The asymptotic behaviour of φ_0 controls the phase-shift modification. The regularity or singularity of φ_0 at the origin controls the modification of singularity of the potential. It also controls the change of Jost function (see section 5.1). Transformations leave the bound spectrum unmodified except for a bound state at energy $\mathcal{E}$ which may be suppressed if it exists or added otherwise. Important properties of the transformations are summarized in table 1. The excluded $n_0=0$ cases lead to transformations of the phase shift which have no simple application.

3. Derivation of phase-equivalent potentials

3.1. Removing a bound state

Before looking at the most general case, let us consider the construction of a phase-equivalent potential with the same bound spectrum as H_0 except that a bound state at energy E_0 is

removed. To this end we perform two successive transformations. The first one suppresses the bound state but modifies the phase shift. The second one restores the phase shift while keeping the bound spectrum unchanged. Both transformations are performed with the same factorization energy, equal to the bound-state energy.

For the first supersymmetric transformation, we choose a bound-state energy $E_0 < 0$ as factorization energy $\mathcal{E}$ and the physical wavefunction $\psi_0^{(0)}$ of this bound state as φ_0,

$$H_0\psi_0^{(0)} = E_0\psi_0^{(0)}. \tag{3.1}$$

Note that this state will not necessarily be the ground state in the two-step algorithm, contrary to what is stated in [15–17]. It follows the behaviours (2.5) and (2.7). According to (2.14), the transformed potential in Hamiltonian H_1 reads

$$V_1 = V_0 - 2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\psi_0^{(0)}. \tag{3.2}$$

The singularity parameter n_1 has increased by one unit. Solutions are given by (2.19). At energy E_0, the solution regular at the origin reads with $r_0 = \beta = 0$ as

$$\varphi_1^{(0)} = \left(\psi_0^{(0)}\right)^{-1}\int_0^r \left(\psi_0^{(0)}\right)^2 \mathrm{d}t. \tag{3.3}$$

It is now non-physical, since its asymptotic behaviour at infinity is $\exp(+\sqrt{-E_0}r)$.

Following [15], let us perform a second supersymmetric transformation with the same factorization energy to restore the phase shift. To this end, an asymptotically increasing solution must be employed. Choosing $\varphi_1^{(0)}$ leaves the spectrum unchanged. Hamiltonian H_1 is factorized as

$$H_1 = A_1^+A_1^- + E_0. \tag{3.4}$$

This Hamiltonian has no bound state at E_0 so that now the factorization energy is not any more a bound-state energy. The first-order operators involve the non-physical solution $\varphi_1^{(0)}$,

$$A_1^{\pm} = \pm\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\mathrm{d}}{\mathrm{d}r}\ln\varphi_1^{(0)}. \tag{3.5}$$

The supersymmetric partner of H_1 is

$$H_2 = A_1^-A_1^+ + E_0. \tag{3.6}$$

The corresponding potential reads

$$V_2 = V_1 - 2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\varphi_1^{(0)} \tag{3.7}$$

$$= V_0 - 2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\int_0^r \left(\psi_0^{(0)}\right)^2 \mathrm{d}t. \tag{3.8}$$

Hence according to table 1, the phase shift is restored,

$$\delta_2(k) = \delta_0(k). \tag{3.9}$$

Potential V_2 is phase-equivalent to V_0. Note that (3.8) provides a direct calculation of V_2 from V_0. Moreover the denominator in (3.8) cannot vanish for $r > 0$, even when $\psi_0^{(0)}$ has nodes. In [23], phase equivalence is discussed with the variable-phase method.

The wavefunctions of H_2 are given by

$$\psi_2 = \psi_0 - \psi_0^{(0)}\frac{\int_0^r \psi_0^{(0)}\psi_0\,\mathrm{d}t}{\int_0^r \left(\psi_0^{(0)}\right)^2\mathrm{d}t}. \tag{3.10}$$

The removed bound state is not restored by the second transformation because $\varphi_1^{(0)}$ is not singular at the origin. Since $\psi_0^{(0)}$ decreases exponentially at large distances, the asymptotic behaviours of ψ_2 and ψ_0 are clearly identical. This explains how the phase shifts can be identical at all energies. Finally the potential singularity at the origin is modified as

$$V_2(r) \underset{r\to 0}{\to} (n_0+2)(n_0+3)r^{-2}. \tag{3.11}$$

The new singularity parameter is $n_2 = n_0 + 2$.

In a strict sense, the previous derivation is valid for the ground state only since the intermediate potential V_1 has singularities at the nodes of $\psi_0^{(0)}$ otherwise. However, it has been shown rigorously in [18] that exactly the same expressions (3.8) to (3.10) are obtained when $\psi_0^{(0)}$ is an excited state. Hence, one can consider that the occurrence of potential singularities in the intermediate step of a pair of supersymmetric transformations with the same factorization energy does not affect the final result.

3.2. General phase-equivalent pair of transformations

The different interesting cases (including the case just developed) can be treated with a single unified notation [20]. Let us consider a solution $\varphi_0^{(0)}$ of (2.1) at energy $\mathcal{E} < 0$ with the asymptotic behaviour

$$\varphi_0^{(0)}(r) \underset{r\to\infty}{\to} \exp(-\sqrt{-\mathcal{E}}r). \tag{3.12}$$

This solution corresponds to a bound state if $\mathcal{E}$ is physical or is singular at the origin if $\mathcal{E}$ is non-physical. The initial Hamiltonian is factorized as in (2.10) with A_0^- given by (2.11). The potential V_1 of (2.14) appears in the supersymmetric partner of H_0. The solution at energy $\mathcal{E}$ behaving at infinity as $\exp(+\sqrt{-\mathcal{E}}r)$ is given, up to a sign, by (2.19) with $r_0 = \infty$ as

$$\varphi_1^{(0)} = \left(\varphi_0^{(0)}\right)^{-1}\left[\beta + \int_r^\infty \left(\varphi_0^{(0)}\right)^2 \mathrm{d}t\right] \tag{3.13}$$

where β is a non-vanishing parameter.

Then H_1 is factorized as in (3.4). The operators A_1^+ and A_1^- are given by [16, 20]

$$A_1^\pm = \pm\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\mathrm{d}}{\mathrm{d}r}\ln\left\{\left(\varphi_0^{(0)}\right)^{-1}\left[\beta + \int_r^\infty \left(\varphi_0^{(0)}\right)^2 \mathrm{d}t\right]\right\}. \tag{3.14}$$

Since $\varphi_1^{(0)}$ increases exponentially, the phase shift is corrected by the second supersymmetric transformation. The potential in Hamiltonian H_2 reads

$$V_2 = V_0 - 2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\left[\beta + \int_r^\infty \left(\varphi_0^{(0)}\right)^2 \mathrm{d}t\right]. \tag{3.15}$$

This potential is phase-equivalent to V_0 and has no singularity at finite distances [18], provided that β satisfies the conditions given below.

Three interesting cases can be considered.

(1) If $\mathcal{E}$ and $\varphi_0^{(0)}$ are physical with $\beta = -1$, the bound state is suppressed (section 3.1).
(2) If $\mathcal{E}$ and $\varphi_0^{(0)}$ are non-physical with $\beta > 0$, a bound state is added at energy $\mathcal{E}$ and a parameter appears in the potential.
(3) If $\mathcal{E}$ and $\varphi_0^{(0)}$ are physical with $\beta < -1$ or > 0, the bound spectrum remains unchanged. This is the case considered by Sukumar [11].

Other possible pairs of transformations are either trivial or lead to potentials with singularities at finite distances [16].

Table 2. Properties of phase-equivalent pairs of transformations.

Action	$\mathcal{E}$	$\lim_{r\to 0}\varphi_0^{(0)}$	$\lim_{r\to\infty}\varphi_0^{(0)}$	β	n_2
Suppression	$=E_0$	r^{n_0+1}	$\exp(-\sqrt{-\mathcal{E}}r)$	-1	n_0+2
Addition ($n_0 \geqslant 2$)	$\neq E_0$	r^{-n_0}	$\exp(-\sqrt{-\mathcal{E}}r)$	>0	n_0-2
None	$=E_0$	r^{n_0+1}	$\exp(-\sqrt{-\mathcal{E}}r)$	<-1 or >0	n_0

The fate of the bound state at energy $\mathcal{E}$ differs in the first and third cases according to the value chosen for β. In the first case, the suppressed bound state remains suppressed after the second factorization. In the third case, the second factorization reintroduces a bound state at this energy so that the bound spectrum remains unchanged after the two successive transformations but a parameter is introduced in the potential. This case is obtained in section 3.4 by combining the first two [17].

The potential V_2 can be calculated in a single step with (3.15). As the denominator always keeps the same sign, V_2 is not restricted to cases where $\varphi_0^{(0)}$ is nodeless [18, 19]. The singularity of the potential is modified according to the case: $n_2 = n_0 + 2$ (suppression), $n_2 = n_0 - 2$ (addition) and $n_2 = n_0$ (spectrum unchanged). Note that n_2 must be positive so that the addition requires the condition $n_0 \geqslant 2$. These properties are summarized in table 2.

The corresponding solution at energy E reads

$$\varphi_2 = \varphi_0 - \varphi_0^{(0)}\left[\beta + \int_r^\infty \left(\varphi_0^{(0)}\right)^2 \mathrm{d}t\right]^{-1} \int_r^\infty \varphi_0^{(0)}\varphi_0\,\mathrm{d}t \tag{3.16}$$

(see [20] for the normalization). Let us emphasize that (3.16) is valid at all energies for physical and non-physical solutions of (3.6), bounded at infinity. The fact that the phase shifts are not modified is easily seen in (3.16). Indeed, φ_2 differs from φ_0 by a term which is obviously short-ranged.

3.3. Two apparent objections

The results of the previous paragraph may seem to raise two types of objections. The first one concerns the Levinson theorem. In traditional scattering theory, this theorem [2] states that

$$\delta(0) - \delta(\infty) = N\pi \tag{3.17}$$

where N is the number of bound states of the potential for orbital momentum l. How can one modify the number of bound states without affecting the phase shifts since the difference between the phase shifts at zero and infinite energies are fixed by this theorem? The answer to this objection has been partly given by Swan [24]: the Levinson theorem is modified in the presence of a singularity (2.4) where the singularity parameter n of partial wave l is not equal to l.

Swan's generalization is proved for a short-ranged potential. No proof is available for a Coulomb asymptotic behaviour such as in (2.3). In [25], we conjectured that Swan's result remains valid in the presence of a Coulomb asymptotic behaviour and that the generalized Levinson theorem reads like in [24]

$$\delta(0) - \delta(\infty) = \left[N + \tfrac{1}{2}(n-l)\right]\pi. \tag{3.18}$$

Phase equivalence is possible in removals and additions of bound states because $N + \frac{1}{2}n$ remains constant in such transformations. The interest of (3.18) for nuclear potentials was first raised by Michel and Reidemeister [26].

A second objection concerns the S matrix defined as $S = \exp(2i\delta)$. Phase-equivalent potentials do not modify the S matrix. As is well known however, bound states correspond to poles of the S matrix on the upper imaginary k-axis. The fact that one removes a bound state seems to imply that the corresponding pole is removed in contradiction with the phase-equivalence relation obtained from (3.9) [27],

$$S_2(k) = S_0(k). \tag{3.19}$$

This apparent contradiction is resolved as follows. The S matrix of a singular potential with singularity parameter n is given in terms of its Jost function by

$$S(k) = (-1)^{n-l}\frac{F(-k)}{F(k)}. \tag{3.20}$$

Bound states correspond to zeros of $F(k)$ on the upper half k-axis and hence to poles of S. The modifications of the Jost function by supersymmetric transformations are given by equations (5.9) and (5.10) according to the behaviour (2.7) or (2.8) of $\varphi_0^{(0)}$ at the origin. For the removal of a bound state, for example, the Jost function is modified twice as in (5.9) but with opposite values $\kappa = \pm\sqrt{\mathcal{E}}$. By combining the modifications induced by the pair of transformations, one obtains the modification of the Jost function [25]

$$F_2(k) = F_0(k)\frac{k^2}{k^2 - \mathcal{E}}. \tag{3.21}$$

Introducing relation (3.21) in (3.20) leads to (3.19). A bound state is removed by suppressing a zero of F_0 but a pole appears in F_2 which leads to the same S matrix (see also [28, 29]).

3.4. Most general transformations of a real energy-independent potential

The most general problem concerns the phase-equivalent potential corresponding to an arbitrary transformation of the bound spectrum. Let us, however, stress that possible transformations are restricted by conditions on the addition of bound states.

Let us consider a set of M arbitrary negative energies $\mathcal{E}^{(i)}$ ($i = 1, \ldots, M$). Some of them may be actual bound-state energies of the initial potential V_0. At each of these energies, we wish to suppress an existing bound state or only modify the potential without removing an existing state or add a new bound state at an energy not belonging to the bound spectrum. Since these energies are arbitrary, the order in which they are classified is irrelevant. We construct a succession of M potentials V_{2i}, phase-equivalent to V_0, where some property is modified at energy $\mathcal{E}^{(i)}$. A function $\varphi_0^{(i)}$ and a parameter β_i as defined in table 2 are also associated to modification i. With such a chain of potentials, any modification of the bound spectrum compatible with the restriction on additions can be reached. The behaviour of (3.16) at small r values easily shows that the parameter β_i can modify in a arbitrary way the normalization constant [1] of the bound state at energy $\mathcal{E}^{(i)}$ (if it is not suppressed) without affecting the normalization constants of the other states. Hence the most general form of phase-equivalent potentials can be obtained by iterations of transformation (3.15),

$$V_{2M} = V_0 - 2\frac{\mathrm{d}^2}{\mathrm{d}r^2}\ln\prod_{i=1}^{M}\left[\beta_i + \int_r^\infty \left(\varphi_{2i-2}^{(i)}\right)^2 \mathrm{d}t\right] \tag{3.22}$$

where $\varphi_{2m}^{(i)}$ is the solution of V_{2m} corresponding to $\varphi_0^{(i)}$. This iteration scheme, though numerically convenient, requires solving the Schrödinger equation at each step. A single expression depending only on solutions $\varphi_0^{(i)}$ of the initial equation can also be derived [18–20].

Following [17, 18], an $M \times M$ matrix $X_0^{(M)}$ is defined with elements

$$X_0^{(i,j)} = \beta_i \delta_{ij} + \int_r^\infty \varphi_0^{(i)} \varphi_0^{(j)} \, \mathrm{d}t \tag{3.23}$$

for $i, j = 1, \ldots, M$. The final potential V_{2M} is given by [20]

$$V_{2M} = V_0 - 2 \frac{\mathrm{d}^2}{\mathrm{d}r^2} \ln \det X_0^{(M)}. \tag{3.24}$$

This expression is easily proved by iteration [17, 18, 20]. Particular cases discussed in [17, 18] are unified by this expression. The singularity parameter is modified by twice the difference between the numbers of removed and added bound states. It may not become negative. A compact expression is also available for the wavefunction [20]. In [18], the supersymmetric method is proved to generalize the method of Abraham and Moses [30] based on the traditional formalism (compare also with [31]).

Among the particular cases of (3.24), let us mention two examples with $M = 2$. With two identical energies $\mathcal{E}^{(1)} = \mathcal{E}^{(2)}$ corresponding to a bound state of V_0, one can first remove this state and then reintroduce it at the same energy. The phase-equivalent potential obtained with this procedure is identical to the potential obtained with the third case of (3.15). The final potential contains a free parameter β_2. With the same procedure but where energy $\mathcal{E}^{(2)}$ is different from $\mathcal{E}^{(1)}$, one can move a bound state to a different energy, also with the introduction of a parameter.

3.5. Generalizations to other types of potentials

The most general form of phase-equivalent potentials is available in the case of single-channel scattering with a real energy-independent potential. However, practical calculations may involve complex optical potentials, energy-dependent potentials or multichannel scattering. Supersymmetric transformations can be extended in these directions and a number of steps have already been performed but the same level of generality has not been reached yet.

A first type of generalization concerns complex potentials. Complex optical potentials are useful to simulate the effect of absorption into inelastic and reaction channels in a single-channel framework [32]. Phase-equivalent potentials have been derived for the removal of states in a complex potential [25, 33, 34]. A first difficulty lies in the replacement of the notion of bound state by the notion of normalizable solution. Complex eigenenergies and the corresponding normalizable solutions are difficult to obtain for a complex potential. Moreover solutions corresponding to a positive real part of the energy eigenvalue lead to unphysically oscillating potentials, when removed. Narrow resonances visible on complex phase shifts should not be removed even when they correspond to a normalizable solution [25].

Some potentials possess an energy dependence. This case is more complicated and not very favourable for phase-equivalent supersymmetric transformations. However, the particular case of a linear energy dependence can easily be treated. The expression for the removal of a bound state can be found in [35].

The generalization to multichannel scattering is an interesting but difficult problem. It requires passing to a matrix formalism [36]. However, several new difficulties appear. At some time, it was believed that phase-equivalent potentials could not be constructed with the supersymmetric formalism [28]. However such a construction was realized in [37] for the removal case (see also [38]). Preliminary results in the same direction were obtained in [39]. The addition of bound states can also be treated [40].

A first difficulty of the multichannel phase equivalence arises from the large number of behaviours at 0 and ∞ that multichannel solutions may have. A second related difficulty is

that degenerate bound states exist for multichannel potentials and generally appear when a bound state is added [40].

Let us briefly describe the removal case for a Schrödinger equation describing N coupled channels,

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V_0(r)\right) \Phi_0(r) = E\,\Phi_0(r) \tag{3.25}$$

where all masses are supposed equal for simplicity. The potential V_0 is an $N \times N$ Hermitian matrix and Φ_0 is a column eigenvector. The asymptotic limit of the potential,

$$V_0(r) \underset{r\to\infty}{\to} \Delta \tag{3.26}$$

is a diagonal matrix whose diagonal elements Δ_{ii} are the thresholds of the different channels. Let $\Psi_0^{(0)}$ be a normalized eigenvector at the physical energy E_0,

$$\int_0^\infty \Psi_0^{(0)\dagger}(r)\Psi_0^{(0)}(r)\,\mathrm{d}r = 1$$

where $\Psi_0^{(0)\dagger}$ is the adjoint row vector of $\Psi_0^{(0)}$. A phase-equivalent potential matrix with energy E_0 removed reads

$$V_2(r) = V_0(r) - 2\frac{\mathrm{d}}{\mathrm{d}r}\frac{\Psi_0^{(0)}(r)\Psi_0^{(0)\dagger}(r)}{\int_0^r \Psi_0^{(0)\dagger}(t)\Psi_0^{(0)}(t)\,\mathrm{d}t}. \tag{3.27}$$

This potential provides the same $N \times N$ scattering matrix as V_0. The corresponding eigenvectors are given in [37].

4. Applications of phase equivalence

4.1. Solvable potentials

The general technique described in the previous section can be applied to derive analytical expressions of phase-equivalent potentials in the case of solvable potentials.

The first case studied was the attractive Coulomb potential. Amado [41] derived potentials for each partial wave providing the Coulomb phase shifts but without the lowest bound state. Further potentials phase equivalent to Coulomb were constructed analytically and studied in [17, 42, 43].

Solvable potentials phase equivalent to the Morse and Hulthén potentials have also been obtained [23, 44]. The Pöschl–Teller and generalized Pöschl–Teller potentials are considered in [45, 46]. The Eckart potential is discussed in [47]. Closed algebraic expressions have also been obtained for potentials that are phase equivalent with the generalized Ginocchio potential, which is a member of the Natanzon class [48].

4.2. Deep and shallow potentials

In nuclear physics, the deep or shallow nature of the nucleus–nucleus interaction has been a long-standing problem. Microscopic models favour deep potentials to simulate the effect of the Pauli principle between the interacting nuclei. Deep potentials possess bound states below the physical spectrum that simulate states forbidden by the Pauli exclusion principle between the nucleons constituting these nuclei.

Elastic scattering data do provide both deep and shallow potentials. For instance, $\alpha + \alpha$ scattering can be described equally well with a shallow potential exhibiting a complicated orbital-momentum dependence [49] and by a simple deep potential without such a dependence [50]. Only even partial waves are relevant for two identical bosons. The deep potential

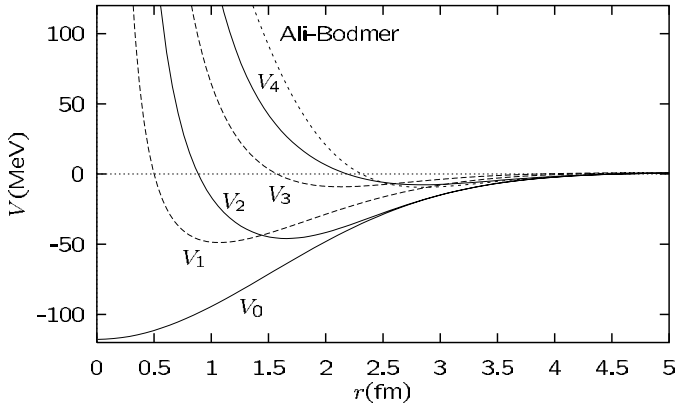

Figure 1. Deep $\alpha + \alpha$ potential [50] for the s wave (V_0) and its successive supersymmetric partners ($V_1, \ldots, V_4$). Phase-equivalent potentials are represented as full lines. The potential of Ali and Bodmer [49] (dotted line) is shown for comparison.

possesses three unphysical bound states (two in the s wave and one in the d wave) which simulate Pauli-forbidden states of microscopic descriptions of this collision (i.e. states of the relative motion that correspond to vanishing solutions). Supersymmetric transformations helped to clarify the problem [15].

Figure 1 shows the effect of removing the two forbidden states in the s wave of the deep $\alpha + \alpha$ potential of Buck *et al* [50]. Potential V_4 is similar to the shallow phenomenological potential of Ali and Bodmer [49]. The shallow potential displays a strong repulsive core qualitatively similar to the singularity of V_4. This resemblance indicates that the deep empirical potential V_0 of Buck *et al* and the shallow empirical potential of Ali and Bodmer are approximately phase equivalent. They cannot be exactly phase equivalent since the different numbers of bound states of these non-singular potentials lead to different Levinson theorems (3.17). Potentials V_2 and V_4 are phase equivalent to V_0 and have respectively one and two bound states less. The intermediate potentials V_1 and V_3 have the same number of bound states as V_2 and V_4 but are not phase equivalent to V_0. Each transformation increases the r^{-2} repulsive singularity at the origin [15]. With equation (3.15) or (3.24), one can obtain V_2 and V_4 in a simple and direct way and avoid calculating V_1 and V_3.

Phase-equivalent supersymmetric transformations have been applied to different systems such as d + n [51], α + n [27, 52], $\alpha + \alpha$ [15, 16, 53], α + ^{14}C [54], α + ^{16}O [33, 35] (see [26] for a very close approximate treatment) and ^{16}O + ^{16}O [25].

For the ^{16}O + ^{16}O elastic scattering, deep and shallow complex optical potentials provide essentially similar fits of experimental excitation functions over the 10 to 35 MeV range of centre-of-mass energies. After removing complex normalizable solutions from a deep optical potential, the real and imaginary parts of the resulting potentials for the $l = 12$ to 20 dominant partial waves resemble the shallow-potential ones [25].

Finally, phase-equivalent transformations have also been used to derive local potentials from resonating-group calculations [55, 56].

4.3. Off-shell effects in reactions

The role of off-shell effects is studied in a number of types of reactions [57]. Usually, these effects are found rather weak with two exceptions that we discuss first.

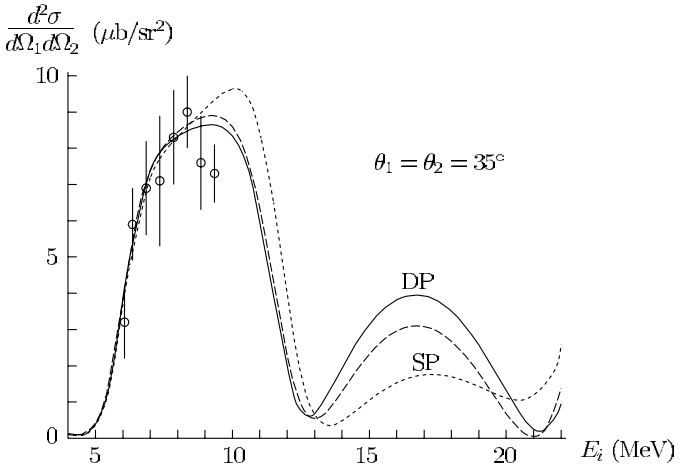

Figure 2. $\alpha(\alpha, \alpha\gamma)\alpha$ experimental cross sections (circles), compared with microscopic (dashed line), deep-potential (DP, full line) and shallow-potential (SP, dotted line) calculations [58]. Experimental data are from [61].

A charged particle accelerated by a Coulomb force emits radiation. This photon-emission process known as bremsstrahlung can also occur in a collision between nuclei because of the acceleration due to the combined effect of the strong nuclear and Coulomb forces. Nucleus–nucleus bremsstrahlung corresponds to a transition between continuum states, the relative motion of the colliding nuclei being slowed down by the emission of the photon. Model calculations involve an electromagnetic transition between initial and final scattering waves with different energies. The $\alpha + \alpha$ bremsstrahlung is dominated by E2 transitions. It has been studied with the resonating-group microscopic model [58, 59] and with the potential model [60] with both the deep potential of [50] and its phase-equivalent shallow potential. For the s waves, they are displayed as V_0 and V_4 in figure 1. For the d waves, only one forbidden state is removed from the deep potential, while for larger l values there is no forbidden bound state [15, 50].

In the available experimental data [61], the α particles have an initial centre-of-mass energy E_i below 10 MeV (see figure 2). After the collision, they are detected in the coplanar directions $\theta_1 = \theta_2 = 35^\circ$ while the emitted photon remains undetected. All theoretical calculations reproduce the data reasonably well. However, the shallow-potential cross section is smaller beyond 13 MeV than the deep-potential one by a factor of 2. Below 13 MeV, the microscopic calculation (dashed line) agrees fairly well with the deep-potential result. Additional data with improved statistics would allow discriminating between the models [58, 59, 62].

In the same spirit, proton–proton bremsstrahlung might be useful to discriminate deep and shallow nucleon–nucleon potentials [63].

Significant off-shell effects are also found in the photodisintegration of ^{7}Li into α and ^{3}H [64]. Here also the results obtained with a deep potential agree better with experiment than those obtained with a shallow potential.

Off-shell effects are weak in a number of other reactions. In the ^{16}O + ^{17}O collision studied with the coupled-reaction-channel model [65], a measurable difference between phase-equivalent potentials only appears at scattering angles around 90°. Unfortunately, data are not available in the interesting energy domain. In breakup reactions [45, 66, 67], these effects are essentially negligible.

Table 3. Ground-state ^{12}C energy and root-mean-square radius in a 3α model. The phase-equivalent potentials V_0 and V_4 are taken as $V_{\alpha\alpha}$. For V_0, a projection technique [70] is used to eliminate the two-body forbidden states.

^{12}C	$V_{\alpha\alpha} = V_0$	$V_{\alpha\alpha} = V_4$	Experiment
E (MeV)	−0.26	−1.01	−7.29
$\sqrt{\langle r^2 \rangle}$ (fm)	2.80	2.80	2.48
λ	1.096	1.088	
E (MeV)	−7.29	−7.29	−7.29
$\sqrt{\langle r^2 \rangle}$ (fm)	2.35	2.46	2.48

4.4. Off-shell effects in bound three-body systems

In reaction studies, deep local potentials often provide better results than shallow potentials. However, for the study of bound states they have an enormous drawback. The non-physical states of the two-body interactions give rise to numerous unphysical states in many-body systems [68]. It is difficult to extract meaningful results from their spectra. Calculations with shallow phase-equivalent potentials avoid this problem [68] and prevent the collapse of three-body systems [69]. Shallow phase-equivalent potentials are easier to use except that the employed numerical technique must be able to deal with an l dependence of the potential.

The deep-potential problem is usually solved by a projection technique which does not affect the nucleus–nucleus potential, but introduces nonlocality in the three-body Hamiltonian [70]. Unphysical two-body bound states are moved to sufficiently high energies so that they do not affect much the three-body spectrum. In such calculations, very accurate numerical conditions are required [71, 72]. The results of this approach can be compared with the supersymmetric elimination of the unphysical bound states [52, 72].

Let us illustrate this with a three-α model of ^{12}C [71, 72]. Its Hamiltonian reads

$$H = -\frac{\hbar^2}{2m_\alpha}(\Delta_1 + \Delta_2 + \Delta_3) + \lambda[V_{\alpha\alpha}(r_{23}) + V_{\alpha\alpha}(r_{31}) + V_{\alpha\alpha}(r_{12})] \tag{4.1}$$

where $\boldsymbol{r}_i$ is the coordinate of α particle i and $\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$ is the relative coordinate between α particles i and j. Different choices are made for the interaction potentials $V_{\alpha\alpha}$ between α particles. The parameter λ is equal to unity for the bare $\alpha\alpha$ interaction. The results for $\lambda = 1$ are shown in the first two rows of table 3. Without the projection technique, the energy would be −240.65 MeV. As can be seen from the table, the deep potential V_0 with elimination of the two-body forbidden states and the phase-equivalent shallow potential V_4 lead to qualitatively similar results. The off-shell effects lead to an energy difference of 0.75 MeV. Since the results are quite far from experiment, this is a poor model for ^{12}C. The validity of the model can also be tested by adjusting λ to fit the experimental ground-state energy (see table 3). The two models still provide different results as shown by the root-mean-square radii. Contrasted densities can be seen in figure 7 of [72]. Note that with the λ values needed to reproduce the ^{12}C ground-state energy, the 2α system ^{8}Be would be bound!

The α + n + p system has been used to determine the α + d scattering length [73] and to study the ^{6}Li nucleus [74]. Phase-equivalent potentials do not lead to very different results. Similar studies were performed on the three-nucleon [75] and $\alpha + \alpha$ + N systems [76].

Light nuclei with a neutron excess and a small binding energy possess much larger radii than stable nuclei. The neutron density distribution displays an extended shape known as a halo. A halo nucleus is viewed as made of a core resembling a normal nucleus surrounded by

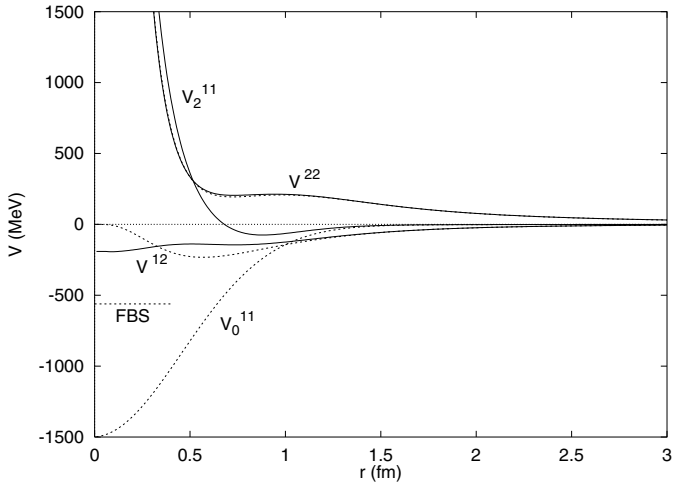

Figure 3. Diagonal and non-diagonal components (1: ^{3}S, 2: ^{3}D) of the Moscow nucleon–nucleon potential [84] (dotted lines) and of the phase-equivalent shallow potential [37] (full lines) after removal of the forbidden bound state (FBS).

one or two loosely bound neutrons. Two-neutron halo nuclei such as ^{6}He, ^{11}Li and ^{14}Be are well described by a three-body model involving core-neutron and neutron–neutron interactions [77]. Occupied neutron orbitals of the core are forbidden to the halo neutrons in the deep core-neutron potential. Halo nuclei are studied in [52, 72, 77–81]. Phase equivalence has been analysed in studies of ^{6}He [52, 72, 78–80], ^{11}Li [52, 81] and ^{14}Be [52].

In [52], approximate variational calculations are performed with the Lagrange-mesh method for these halo nuclei. A deep core-neutron potential and its shallow phase-equivalent potential lead to only slightly different binding energies. The larger binding energy obtained with the shallow potential induces a smaller radius. However, when the theoretical binding energies are fitted to the experimental value by multiplying the potentials by a constant like in (4.1), the radii become essentially the same. In the case of ^{14}Be, the wavefunctions obtained with the deep and shallow potentials are, however, significantly different (see figure 2 in [52]).

Phase equivalence has also been discussed in an $^{16}\mathrm{O} + \alpha + \mathrm{n}$ model of ^{21}Ne [82] and in a study of the pionic decay of the $^{5}_{\Lambda}$He hypernucleus [83].

4.5. Phase-equivalent coupled-channel potentials

The simplest case of coupled channels is provided by the deuteron where the $^3\mathrm{S}_1$–$^3\mathrm{D}_1$ coupling is due to the tensor force. The deep neutron–proton Moscow potential [84] contains an unphysical bound state which can be exactly removed by a pair of supersymmetric transformations [39, 37]. This removal was first studied in an approximate way in [85].

The Moscow potential and its phase-equivalent potential are displayed in figure 3 [37]. The D component of the potential is not affected much by the transformation. In contrast, the S component becomes much more similar to traditional nucleon–nucleon potentials based on meson exchange except that it presents a $2r^{-2}$ singularity near the origin. The coupling potential is also modified. Note that the wavefunctions of both potentials are quite different at small distances. The S component has a node in the deep-potential case whose existence might be addressed experimentally.

5. Inverse scattering

5.1. Supersymmetric transformations with complex factorization energies

As discussed in the introduction, the possible existence of bound states makes solutions of the inverse problem non-unique. A unique potential is obtained when there is no bound state but, in traditional inversion theory, this can only happen when the phase shift for orbital momentum l takes equal values at zero and infinite energies. Otherwise the behaviour of the phase shift reveals bound states through the Levinson theorem (3.17). Before describing inversion techniques, let us consider supersymmetric transformations with complex factorization energies and study the transformations of Jost functions and S matrices [21].

For a complex factorization energy

$$\mathcal{E} = \kappa^2 \neq 0, \tag{5.1}$$

a solution $\varphi_0(\kappa, r)$ of the Schrödinger equation behaves asymptotically as

$$\varphi_0(\kappa, r) \underset{r\to\infty}{\to} \exp\left[-\mathrm{i}\kappa r + \frac{\mathrm{i}}{2} Z_1 Z_2 \kappa^{-1} \ln(2\mathrm{i}\kappa r)\right] + O(r^{-1}). \tag{5.2}$$

Since we are interested in a potential without bound states, we now focus on transformations described in the last two rows of table 1. These transformations which leave the spectrum unchanged have opposite behaviours both at the origin and at infinity. For a solution regular at the origin, κ is chosen to lie in the upper half k plane in order to ensure the appropriate increasing asymptotic behaviour,

$$\varphi_0(\kappa, r) \underset{r\to 0}{\to} r^{n_0+1} \Rightarrow \operatorname{Im}\kappa > 0. \tag{5.3}$$

For a solution irregular at the origin, κ is chosen to lie in the lower half k plane in order to ensure a decreasing asymptotic behaviour,

$$\varphi_0(\kappa, r) \underset{r\to 0}{\to} r^{-n_0} \Rightarrow \operatorname{Im}\kappa < 0. \tag{5.4}$$

Let us recall that n_0 must be larger than 0 in this case to ensure a transformation of a scattering state into another scattering state and to keep a simple relation between the phase shifts.

With the first-order differential operators (2.11), the potential after a supersymmetric transformation reads according to (2.14)

$$V_1 = V_0 - 2\frac{\mathrm{d}}{\mathrm{d}r}\frac{\varphi_0'}{\varphi_0}. \tag{5.5}$$

Equations (5.2) to (5.5) imply that the transformed potential behaves near the origin as (2.16) with n_1 given in table 1 and asymptotically as (2.15),

$$V_1(r) \underset{r\to\infty}{\to} Z_1 Z_2 r^{-1} + [l(l+1) + \mathrm{i}Z_1 Z_2 \kappa^{-1}] r^{-2} + O(r^{-3}). \tag{5.6}$$

When no Coulomb term is present, the difference between V_1 and V_0 decreases exponentially at infinity. The r^{-2} term is then the same in the asymptotic forms of both the initial and final potentials.

Equation (2.17) provides a relation between the scattering states of V_0 and V_1. The regular solution u_0 [1, 2] is defined with the normalization

$$u_0(r) \underset{r\to 0}{\to} \frac{(kr)^{n_0+1}}{(2n_0+1)!!} + O(r^{n_0+2}). \tag{5.7}$$

Its asymptotic behaviour

$$u_0(r) \underset{r\to\infty}{\to} \frac{1}{2\mathrm{i}}\left[F_0(k)\,\mathrm{e}^{-\mathrm{i}(kr-\frac{1}{2}n_0\pi)} - F_0(-k)\,\mathrm{e}^{\mathrm{i}(kr-\frac{1}{2}n_0\pi)}\right] \tag{5.8}$$

Table 4. Modifications of the phase shift, singularity parameter and Jost function for supersymmetric transformations with a complex energy $\mathcal{E} = \kappa^2 \neq 0$.

$\lim_{r\to 0} \varphi_0$	$\mathrm{Im}\,\kappa$	$\delta_1(k) - \delta_0(k)$	n_1	$F_1(k)/F_0(k)$
r^{n_0+1}	>0	$\arctan(k/\mathrm{i}\kappa)$	n_0+1	$k/(k+\kappa)$
r^{-n_0}	<0	$\arctan(k/\mathrm{i}\kappa)$	n_0-1	$(k-\kappa)/k$

defines the Jost function F_0. The transformed function $A_0^- u_0$ is proportional to a regular solution u_1 of H_1, satisfying (5.7) with n_0 replaced by n_1. The asymptotic behaviour of u_1 verifies an equation similar to (5.8) which provides the Jost functions

$$F_1(k) = F_0(k)\frac{k}{k+\kappa} \tag{5.9}$$

for $\varphi_0 \underset{r\to 0}{\to} r^{n_0+1}$ and

$$F_1(k) = F_0(k)\frac{k-\kappa}{k} \tag{5.10}$$

for $\varphi_0 \underset{r\to 0}{\to} r^{-n_0}$. The scattering matrix (3.20) becomes in all cases

$$S_1(k) = S_0(k)\frac{\kappa+k}{\kappa-k}. \tag{5.11}$$

This equation relates the phase shifts of V_0 and V_1 by

$$\delta_1(k) = \delta_0(k) + \arctan(k/\mathrm{i}\kappa). \tag{5.12}$$

The transformed phase shift is in general complex.

A supersymmetric transformation thus introduces an S-matrix pole at κ. The main characteristics of transformations are summarized in table 4. One can verify that the phase-shift modification is in agreement with the generalized Levinson theorem (3.18) and with the modification of singularity (2.16) at the origin.

5.2. Derivation of the unique singular potential without bound state at fixed l

The results of section 5.1 suggest the following approach to the inverse problem: from a phase shift at fixed l, one determines a singular potential without any bound state. The singularity parameter of the repulsive core of this potential is, according to (3.18) with $N = 0$,

$$n = l + \frac{2}{\pi}[\delta(0) - \delta(\infty)]. \tag{5.13}$$

The singularity parameters of different partial waves are not necessarily equal.

Because of the lack of bound state, this potential is unique. The inverse problem can thus be solved in two steps: (i) determine the unique singular potential, which only depends on scattering data, (ii) add chosen bound states with chosen normalization constants to the spectrum without modifying the phase shifts. Two parameters are thus introduced at each such step.

The second step is explained in section 3.4. The first step can be performed in an approximate way by using a Bargmann-type approximation of the S matrix,

$$S_M(k) = S_0(k)\prod_{m=1}^{M}\frac{\kappa_m+k}{\kappa_m-k}. \tag{5.14}$$

This expression is however not purely Bargmann because of factor S_0 which is the S matrix of a reference potential V_0. For a real potential, the S matrix has to be unitary. This is the case if and only if the poles are symmetric with respect to the imaginary k-axis. The corresponding phase shift is given by

$$\delta_M(k) = \delta_0(k) + \sum_{m=1}^{M} \arctan(k/\mathrm{i}\kappa_m). \tag{5.15}$$

The parameters of any of these expressions are obtained with a fit to given data.

Starting from V_0, the first factor involving κ_1 in the product leads to a potential V_1 by using (5.5). This process is iterated. The factor with κ_2 gives a new potential V_2, and so on.

For each factorization energy $\mathcal{E}_m = \kappa_m^2$, $\varphi_{m-1}(\kappa_m, r)$ is the chosen factorization eigensolution of H_{m-1} and $\varphi_0(\kappa_m, r)$ is the same-type eigensolution of the initial Hamiltonian H_0. The final potential reads, by repeated application of (5.5),

$$V_M(r) = V_0(r) - 2\frac{\mathrm{d}}{\mathrm{d}r} \sum_{m=1}^{M} \frac{\varphi'_{m-1}(\kappa_m, r)}{\varphi_{m-1}(\kappa_m, r)}. \tag{5.16}$$

It can be expressed in terms of solutions from the initial equation [86–88] as

$$V_M(r) = V_0(r) - 2\frac{\mathrm{d}}{\mathrm{d}r} \frac{W'[\varphi_0(\kappa_1, r), \ldots, \varphi_0(\kappa_M, r)]}{W[\varphi_0(\kappa_1, r), \ldots, \varphi_0(\kappa_M, r)]} \tag{5.17}$$

where W is the Wronskian of the different solutions and W' is its derivative with respect to r. The solutions of the corresponding Schrödinger equation read

$$\varphi_M(k, r) = \frac{W[\varphi_0(k, r), \varphi_0(\kappa_1, r), \ldots, \varphi_0(\kappa_M, r)]}{W[\varphi_0(\kappa_1, r), \ldots, \varphi_0(\kappa_M, r)]}. \tag{5.18}$$

Formulae involving determinants of integral forms of 2×2 Wronskians can reduce the sizes of determinants by about a factor of two [89].

The asymptotic difference between the potentials is, by repeated application of (5.6),

$$V_M(r) - V_0(r) \underset{r\to\infty}{\to} \mathrm{i}Z_1Z_2 \sum_{m=1}^{M} \kappa_m^{-1} r^{-2} + O(r^{-3}). \tag{5.19}$$

Since the poles are symmetric with respect to the imaginary k-axis, the first term appearing on the right-hand side of this equation is purely real. As we are interested in potentials decreasing faster than r^{-2} at infinity, this term should vanish. This imposes a condition on the κ_m. Of course, this term disappears when the potential contains no Coulomb term. Moreover, when the initial potential is short-ranged, i.e. decreasing exponentially at infinity, the transformed potentials are also decreasing exponentially. Consequently, the last term of (5.19) decreases exponentially rather than as r^{-3}, and the initial and final potentials are both short-ranged.

An inversion algorithm thus reads as follows. First a potential V_0 is selected which provides the initial scattering matrix S_0 or phase shift δ_0. For example, the choice $V_0 = l(l+1)/r^2$ is convenient when $Z_1Z_2 = 0$ but does not lead to a fast convergence when $Z_1Z_2 \neq 0$. Secondly, the S matrix is fitted with an expression $S_M(k)$ [(5.14)] or the phase shift is fitted with an expression $\delta_M(k)$ [(5.15)]. These fits provide values of M and of the complex parameters κ_m. Let M_+ be the number of poles with $\mathrm{Im}\,\kappa_m > 0$ and M_- be the number of poles with $\mathrm{Im}\,\kappa_m < 0$. The phase shift verifies

$$\delta(0) - \delta(\infty) = \tfrac{1}{2}(M_+ - M_-)\pi. \tag{5.20}$$

With (5.13) and the condition $n \geqslant 0$, this leads to the condition

$$M_+ \geqslant M_- + l. \tag{5.21}$$

Thirdly, the potential V_M is determined with (5.16) or (5.17). Finally, up to

$$N = \tfrac{1}{2}(M_+ - M_- + l) \tag{5.22}$$

bound states can be added if this presents a physical interest.

In principle this technique should be generalizable to multichannel inversion. However, a number of open problems still remain. A typical difficulty arises from the fact that if one starts from an uncoupled initial potential, the transformed potentials are also uncoupled. For example the simple Cox potential [90] could not until now be derived through supersymmetric inversion.

5.3. Inversion at fixed l with real factorization energies

Recently, Samsonov and Stancu [91] have proposed a variant of the supersymmetric approach to the inversion problem. The main differences are: (i) they use real energies only and (ii) they perform the two steps in a single formalism. In addition, they fit their parameters on the phase shifts (5.15) rather than on the S matrix (5.14) as in [21]. The advantage of real over complex energies might be an elimination of weak oscillations in the asymptotic part of the potential. However, complex energies are needed for describing resonances and might be required to approximate effective range expansions in the Coulomb case.

In order to compare both techniques, let us first write the Jost function corresponding to the inversion technique described in section 5.2 as

$$F_M(k) = F_0(k) \prod_{m=1}^{M_+} \frac{k}{k+\kappa_m^+} \prod_{m=M_++1}^{M} \frac{k-\kappa_m^-}{k} \tag{5.23}$$

where we now assume that the first M_+ poles κ_m^+ are located in the upper half k plane and the remaining M_- poles κ_m^- are in the lower half k plane. The Jost function F_0 corresponds to the reference potential V_0. In [91], the Jost function is parametrized in a different way as

$$F_M(k) = F_0(k) \prod_{m=1}^{M_b} \frac{k}{k+\mathrm{i}b_m} \prod_{m=M_b+1}^{M} \frac{k-\mathrm{i}a_m}{k} \tag{5.24}$$

where the M_a coefficients a_m are real and not necessarily positive and the M_b coefficients b_m ($M_b > M_a$) are positive. The imaginary poles $-\mathrm{i}b_m$ of the Jost function are not related to bound states nor to virtual states.

When all a_m are negative, they correspond to virtual states only. Then (5.24) is a particular case of (5.23) and the two methods lead to exactly the same potential. The coefficients κ_m^- and κ_m^+ particularize to $\mathrm{i}a_m$ and $\mathrm{i}b_m$, respectively. However, Samsonov and Stancu also use positive values for a_m. They interpret an imaginary pole of the S matrix in the upper k plane either as a pole or as a zero of the Jost function. The latter values corresponding to bound states, solutions of the second row of table 1 (addition of a bound state), can then be used, which contain a parameter. The two steps of the previous paragraph are combined in a single one. The bound-state locations are however imposed by the fit (5.24). They may eventually be modified with the techniques of section 3.4.

In [92], the same authors propose to use zero-energy solutions in order to modify the asymptotic behaviour (2.3) of the initial potential. Indeed, in the absence of Coulomb term

($Z_1 Z_2 = 0$), the asymptotic behaviour (5.2) is replaced by

$$\varphi_0(0,r) \underset{r\to\infty}{\to} \begin{cases} r^{l+1} + O(r^l) \\ r^{-l} + O(r^{-l-1}). \end{cases} \tag{5.25}$$

The asymptotic behaviour of the transformed potential then becomes

$$V_1(r) \underset{r\to\infty}{\to} V_0(r) + \begin{cases} 2(l+1)r^{-2} + O(r^{-3}) \\ -2lr^{-2} + O(r^{-3}) \end{cases} \tag{5.26}$$

depending on the choice for $\varphi_0(0,r)$. This allows obtaining the correct asymptotic behaviour of an $l \neq 0$ potential starting for example from $V_0 = 0$ but it is only valid in the absence of Coulomb interaction.

The parameter appearing in a zero-energy solution which is singular at both zero and infinity can be adjusted to obtain a correct effective-range expansion [92].

5.4. Inversion at fixed energy

The problem of the inversion at a fixed energy is quite different. The data are a countable set of phase shifts for all partial waves at this energy. Only a limited number of these phase shifts are significantly different from zero. This problem has first been solved by Newton [93]. A supersymmetric approach is also possible.

The Schrödinger equation (2.1) at a given positive energy $E = k^2$ is written as

$$\left[-r\frac{\mathrm{d}^2}{\mathrm{d}r^2}r + r^2(V_0 - E)\right]\chi_0 = -l(l+1)\chi_0 \tag{5.27}$$

after posing $\psi_0 = r\chi_0$. Unlike in previous sections, we separate the centrifugal term from V_0. We consider a particular solution $\varphi_0(l_0, r)$ corresponding to a complex eigenvalue $l_0(l_0+1)$, with the convention $\mathrm{Re}\, l_0 \geqslant -\frac{1}{2}$. Equation (5.27) can be rewritten in factorized form as

$$\left[A_0^+ A_0^- - l_0(l_0+1)\right]\chi_0 = -l(l+1)\chi_0 \tag{5.28}$$

where $l_0(l_0+1)$ plays the role of a factorization constant. The operators A_0^+ and A_0^- are defined as

$$A_0^+ = \left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\varphi_0'}{\varphi_0}\right) r \tag{5.29}$$

and

$$A_0^- = r\left(-\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\varphi_0'}{\varphi_0}\right). \tag{5.30}$$

With the standard permutation one obtains a new equation with the potential

$$V_1 = V_0 - \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} r \frac{\varphi_0'}{\varphi_0}. \tag{5.31}$$

The potential will present singularities if φ_0 is allowed to vanish. Hence we focus on purely ingoing and outgoing solutions with asymptotic behaviour

$$\varphi_0(l_0, r) \underset{r\to\infty}{\to} r^{-1}\left[1 \mp \frac{l_0(l_0+1) + \eta^2 \mp \mathrm{i}\eta}{2\mathrm{i}kr}\right] \exp[\pm\mathrm{i}(kr - \eta \ln 2kr)] \tag{5.32}$$

with $\eta = Z_1 Z_2 / 2k$.

As before, the eigenfunction of V_1 with the eigenvalue $l(l+1)$ is given by

$$\chi_1 = A_0^- \chi_0. \tag{5.33}$$

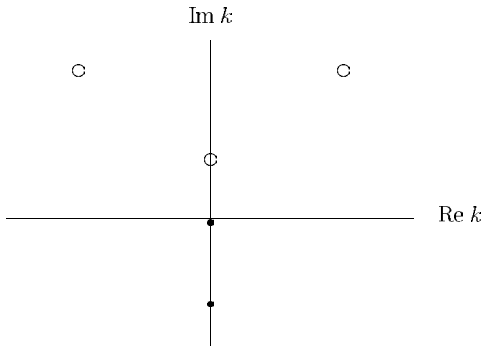

Figure 4. Zeros (dots) and poles (circles) of the Jost function from a fit of $^1S_0$ np phase shifts [21].

From its asymptotic expression, one obtains the transformed S matrix

$$S_1(l) = S_0(l)[l(l+1) - l_0(l_0+1)] \tag{5.34}$$

when φ_0 is purely outgoing or as

$$S_1(l) = S_0(l)[l(l+1) - l_0(l_0+1)]^{-1} \tag{5.35}$$

when φ_0 is purely ingoing. The potential V_1 is, however, complex with an unphysical asymptotic behaviour.

To obtain a real potential, one has to consider pairs of transformations. A first transformation is performed with an outgoing solution corresponding to index l_0. A second transformation is then performed with an ingoing solution corresponding to the conjugate index l_0^*. The resulting transformation of the S matrix is given by

$$S_2(l) = S_0(l)\frac{l(l+1) - l_0(l_0+1)}{l(l+1) - l_0^*(l_0^*+1)}. \tag{5.36}$$

Unitarity of S_0 implies unitarity of S_2.

Equation (5.36) can be iterated M times. The resulting parametrization of the S matrix can be used to determine M parameters l_{2m-2}. This provides the basis for an inversion scheme of the same type as the method based on (5.14). This inversion method is equivalent to the simplest method introduced by Lipperheide and Fiedeldey [94, 95].

Schnizer and Leeb [96, 97] have constructed a general formalism unifying fixed-orbital-momentum and fixed-energy problems, based on generalized Darboux transformations.

6. Applications of supersymmetric inversion

6.1. Nucleon–nucleon scattering

A first example is the inversion of the $^1S_0$ nucleon–nucleon phase shift [21]. A good fit of experimental np phase shifts can be obtained with only five S-matrix poles. These poles are located as shown in figure 4. Three poles in the upper k plane ($M^+ = 3$) lead to three transformations of type (5.3). These are poles of the Jost function and as such non-physical: the pole on the imaginary axis does not correspond to a bound state. Two poles in the lower k plane ($M^- = 2$) lead to two transformations of type (5.4). They correspond to zeros of the Jost function, namely here virtual states. The singularity parameter is $n = 1$. The fact that n is odd leads to an unexpected $2r^{-2}$ singularity. The difference $\delta(0) - \delta(\infty)$ then takes the unusual value $\frac{1}{2}\pi$. The potential is shown as 'np' in figure 5. The curve labelled 'np reg'

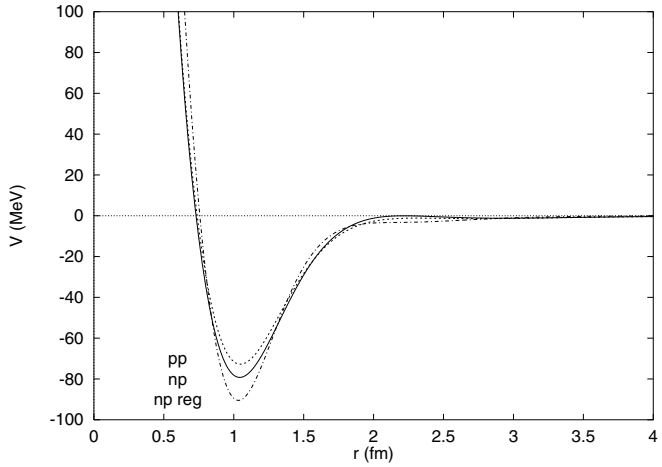

Figure 5. Potentials for $^1S_0$ nucleon–nucleon scattering: singular proton–proton (pp) and neutron–proton (np) potentials of [21] and regular potential of [98] (np reg).

corresponds to the regular potential of [98] with $M_+ = M_- = 3$ but with a pole at -71i whose role is only to regularize the potential and to ensure consistency with the standard Levinson theorem (3.17). Its precise location is not important as long as it is far below the real k-axis. A similar inversion of experimental pp phase shifts is obtained with seven poles, i.e. $M^+ = 4$ and $M^- = 3$ ($n = 1$). The potential is shown as 'pp' in figure 5.

Samsonov and Stancu fit $^1S_0$ experimental np phase shifts with six S-matrix poles on the imaginary axis in (5.24) ($M_a = M_b = 3$) [91]. They simultaneously reproduce the effective range expansion. Their potential does not present a second shallow minimum like the np potential in figure 5 and is very close to experimental soft-core potentials. Four poles being on the upper imaginary k-axis, the singularity parameter is $n = 2$ which corresponds to $\delta(0) - \delta(\infty) = \pi$ and one of the poles can be transformed into a deeply bound state. The same authors fit the $^1P_1$ and $^1D_2$ np phase shifts with six and four poles, respectively [92]. They use parameters in zero-energy solutions to obtain correct effective-range expansions in both partial waves.

6.2. Searches for l-independent potentials

For a given partial wave, the supersymmetric inversion method described above leads to the complete family of potentials reproducing the experimental phase shifts. A number of bound states (with a maximum given by (5.22)) can be added to the singular inversion potential, with arbitrary energies and normalization constants. When the system possesses physical bound states, the potential can thus be made to reproduce the experimental spectrum [100].

As discussed in section 4.2, the simplest potentials describing the interaction between composite particles are often deep and l independent. When such a potential exists, i.e. when experimental phase shifts are compatible with an l-independent potential, this potential can be constructed with the inversion technique presented above. For a given partial wave, an inversion potential with physical bound states is first constructed. Then, like physical bound states, forbidden states are added to this inversion potential without modifying the phase shifts. In contrast to physical bound states, their energies and normalization constants cannot be related to experimental data. However, these energies and normalization constants can be

considered as free parameters that can be varied in order to fit phase shifts of partial waves different from the inverted one. This procedure assumes that the central potential is the same for all partial waves. If such a fit is possible, this assumption is found to be valid and the required potential is obtained. If such a fit is not possible, it proves that the experimental phase shifts are not compatible with an l-independent potential since the inversion technique gives access to the complete family of potentials reproducing the phase shifts of the inverted partial wave.

This procedure has been applied to various systems in a simplified form: a potential without bound state is constructed by supersymmetric inversion, physical states are added with phase-equivalent supersymmetric pairs and Pauli forbidden states are added phenomenologically by regularizing the obtained potential at short distances with an attractive parabolic core. This last step simplifies the parameter search and suppresses numerical instabilities in the short-distance behaviour of the potential constructed by supersymmetric transformations. In [22], this method has been used for the $\alpha + \alpha$ system, for which a deep l-independent potential very close to that of [50] has been constructed from the scattering phase shifts. It has also led to new l-independent potentials for the scattering of electrons on noble-gas atoms [22]. With the same technique, an important simplification of the neutron–proton interaction (no parity and spin dependence) is obtained [99], as compared with the deep Moscow potential [84]. This simplification is only possible for a deep potential, in contrast with usual shallow nucleon–nucleon interactions.

Recently, this technique has for instance been applied to construct $^{12}\mathrm{C} + \alpha$ potentials that reproduce both the experimental elastic-scattering phase shifts and some bound states in the $^{16}\mathrm{O}$ spectrum [100]. The $^{12}\mathrm{C} + \alpha$ system is described by a new l-independent potential which also reproduces the physical properties of the whole rotational band built on the 0^+_2 excited state. This has led to a prediction for the normalization constant of the 2^+_1 subthreshold state in the $^{16}\mathrm{O}$ spectrum which has a strong impact on the $^{12}\mathrm{C}(\alpha,\gamma)^{16}\mathrm{O}$ capture cross section in nuclear astrophysics.

7. Conclusion and outlook

Supersymmetric quantum mechanics has brought a fresh look at the inverse problem. The introduction of singular potentials in the theory has opened a way to new applications. Moreover supersymmetric transformations often provide a strong simplification with respect to the traditional approach.

In the particular phase-equivalence problem, pairs of supersymmetric transformations allow the construction of exactly phase-equivalent potentials with arbitrarily different bound spectra (subjected to an unavoidable condition on the maximum number of bound states that can be added). This technique has been extended from real potentials to complex, linearly energy-dependent and multichannel potentials.

The ambiguity between deep and shallow potentials can be resolved in an intuitive way. This approach has given rise to many studies on the physical influence of the non-physical bound states appearing in inversion techniques. In many cases, this influence is rather weak but some reactions such as bremsstrahlung should allow physically discriminating between deep and shallow potentials.

The genuine inverse problem, i.e. deriving a potential from scattering data can be solved in a simple approximate way with supersymmetry. The construction of a unique singular potential without any bound state opens a way to variants of the traditional approach. By adding bound states, one can for example search for l-independent potentials.

Several problems remain open. The most difficult one is the inverse coupled-channel scattering. It is even more difficult in the presence of thresholds.

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Appendix. Inversion with the Marchenko equation

The standard approach to the inverse scattering problem at fixed l is based on the Gel'fand–Levitan or Marchenko equations [1–3]. Here we briefly summarize the Marchenko approach which is based on the scattering matrix and is therefore closer to the supersymmetric technique described in section 5. The method with the Gel'fand–Levitan equation is based on the knowledge of the Jost function.

From the scattering matrix $S_l(k) = \exp[2\mathrm{i}\delta_l(k)]$ in the absence of bound states, one calculates [1, 98, 101]

$$F_l(r,t) = -(2\pi)^{-1}\int_{-\infty}^{\infty} h_l(kr)[S_l(k)-1]h_l(kt)\,\mathrm{d}k \tag{A.1}$$

where h_l is a Riccati–Hankel function. For example $h_0(kr) = \exp(\mathrm{i}kr)$. The Marchenko equation reads

$$A_l(r,t) + F_l(r,t) + \int_r^{\infty} A_l(r,s)F_l(s,t)\,\mathrm{d}s = 0. \tag{A.2}$$

After calculation of the kernel $A_l(r,t)$, the potential is given by

$$V_l(r) = -2\frac{\mathrm{d}}{\mathrm{d}r}A_l(r,r). \tag{A.3}$$

When the S matrix is approximated by the rational expression (5.14) but with $S_0 = 1$, the integral in (A.1) can be performed algebraically. This case is then equivalent to the supersymmetric inversion of section 5.2 [34].

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